

Transfer matrix for Kogut-Susskind fermions in the spin basis

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Abstract

In the absence of interaction it is well known that the Kogut-Susskind regularizations of fermions in the spin and flavor basis are equivalent to each other. In this paper we clarify the difference between the two formulations in the presence of interaction with gauge fields. We then derive an explicit expression of the transfer matrix in the spin basis by a unitary transformation on that one in the flavor basis which is known. The essential key ingredient is the explicit construction of the fermion Fock space for variables which live on blocks. Therefore the transfer matrix generates time translations of two lattice units.

1 Introduction

The naive discretization of the Dirac equation on the lattice [1] leads to a replication of the fermionic states, known as lattice fermion *doubling* [2]. The doublers appear as spurious poles in the fermion propagator at the nonzero corners of the Brillouin zone. The Wilson way of removing the doublers is to give them a mass which becomes infinite in the continuum limit, at the cost of an explicit breaking of chiral invariance on the lattice [2].

In the Kogut-Susskind [3–6] lattice formulation for relativistic fermions [7, Chap. 4] the doublers are instead interpreted as physical fields by the introduction of additional quantum numbers. This has been done in two ways. In the former approach, first the fermion field is reduced to a single component per site by a procedure called *spin diagonalization*, and, for this reason, this method is referred to as the one in the *spin basis*. Afterwards spin and flavour degrees of freedom are associated to different corners of an elementary hypercube on the lattice [8–10], and therefore sometimes fermions in this formulation are said to be *staggered*. In the latter approach [11–13], said in the *flavour basis*, the additional quantum numbers, called *taste*, are associated, together with the spin, with blocks corresponding to the hypercubes of the spin basis of size twice the lattice spacing.

In the absence of coupling with gauge fields these forms are changed into one another by a linear transformation on the fermion fields, but in the presence of gauge fields they are not equivalent, as we shall make clear in the following. Their difference is of consequence in the construction of the corresponding transfer matrices.

For Kogut-Susskind fermions in the flavour basis a simple operator realization of the transfer matrix is known [14]. It has been built in close analogy with the case of Wilson fermions [15–19] (see also [20]), the only difference being that it performs time translations by one block instead of one lattice spacing.

The situation is more complex for Kogut-Susskind fermions in the spin basis [11, 12, 21], because all attempts at constructing a positive definite transfer matrix that performs time translations by a single lattice spacing failed. The difficulty was circumvented by looking at time translations by two lattice spacings. Here we meet with a subtlety. We must distinguish whether the Fock space is built on one or two time slices. In the first case, the square root of the matrix which translates by two lattice spacings is the one that translates by one lattice spacing. In the second case, instead, translations by one lattice spacing are not defined at all. This seems to be the case with Kogut-Susskind fermions, but the necessary construction of the Fock space on blocks, in the spin basis, has not been made explicit.

We became interested in a formulation of the transfer matrix in the spin basis in the framework of relativistic field theories of fermions whose partition function is

dominated by bosonic composites [22]. This subject became for us more relevant in the development of an approach to QCD hadronization (meant as the replacement of the QCD degrees of freedom by hadronic ones) that makes use of the operator form of the transfer matrix [23–26]. Using Kogut-Susskind fermions, because of the lack of a convenient formulation of the transfer matrix, we were able to express our results only in the flavour basis. Numerical simulations are, instead, usually performed in the spin basis, because they are much faster. We were thus motivated to find an operator form of the transfer matrix in this latter basis as well. Since apparently in any case we should resign to time translations by one block, we decided to get an expression of the transfer matrix in the spin basis by a linear transformation from the flavour basis.

We deem that the question might be of more general interest, and therefore we report our results in the present paper. In Sect. 2 we remind for the convenience of the reader and in order to establish the notation what is relevant for the following about the Kogut-Susskind regularization. We adopt the notations of Montvay and Münster [7] with some minor changes that will be specified. In Sect. 3 we perform the transformation of the action from the flavour to the spin basis. Most of the results, with some qualification, are well known, but we think this Section is a necessary preparation for Sect. 4, in which we perform the transformation of the transfer matrix.

2 Kogut-Susskind fermions

Let x_μ be the coordinates of hypercubic lattice sites, $0 \leq x_\mu \leq L_\mu - 1$, $0 \leq \mu \leq 3$ (Montvay and Münster in [7] use indices from 1 to 4), and y_μ the coordinates of hypercubic blocks. They are related by

$$x_\mu = 2y_\mu + \eta_\mu \quad (1)$$

with $0 \leq y_\mu \leq L'_\mu - 1$, $L_\mu = 2L'_\mu$, and $\eta_\mu = 0, 1$ the position vectors within the block. The sum over lattice points can be split into the sum over the blocks and the sum over the sites within a block, that is

$$\sum_x = \sum_y \sum_\eta . \quad (2)$$

We denote by ψ_x the fermionic fields on the lattice sites, and by $q_y^{\alpha a}$ the fields on the blocks. The latter have Dirac spinor indices $1 \leq \alpha \leq 4$, in greek letters, and taste indices $1 \leq a \leq 4$, in latin letters.

It is important to remark that the gauge transformations in the first case act at the sites of the basic lattice, in the second at the coordinates of the blocks

$$\psi_x \rightarrow g_x \psi_x , \quad q_y^{\alpha a} \rightarrow g_y q_y^{\alpha a} . \quad (3)$$

While g_y is the same transformation for all x in a given block with coordinate y , g_x will in general change also within the same block.

2.1 The flavour basis

The gauge link variables on the blocks are denoted by $U_\mu(y)$. Under gauge transformations they change according to the rule

$$U_\mu(y) \rightarrow g_y U_\mu(y) g_{y+\hat{\mu}}^\dagger. \quad (4)$$

The action of the fermion fields in the flavour basis can be written as

$$S(U) = 2^4 \sum_y \mathcal{L}_q(U) \quad (5)$$

where the factor 16 keeps into account the volume of the elementary cell when using variables defined on the blocks, and the Lagrangian in the flavour basis is

$$\begin{aligned} \mathcal{L}_q(U) &:= m \bar{q}_y (\mathbb{1} \otimes \mathbb{1}) q_y \\ &+ \sum_{\mu=0}^3 \bar{q}_y \left\{ \left[(\gamma_\mu \otimes \mathbb{1}) \frac{1}{2} (\nabla_\mu^{(+)} + \nabla_\mu^{(-)}) - (\gamma_5 \otimes t_5 t_\mu) \Delta_\mu \right] q \right\}_y \end{aligned} \quad (6)$$

the flavour matrices t_μ are defined for $\mu = 0, \dots, 3$ and $\mu = 5$ by

$$t_\mu = \gamma_\mu^T := t_\mu^\dagger \quad (7)$$

and the other operators are defined in terms of translations on the blocks

$$[T_\mu^{(\pm)} f]_y := 2^4 \sum_{y'} \frac{1}{2^4} \delta_{y', y \pm \hat{\mu}} f(y') = f(y \pm \hat{\mu}) \quad (8)$$

and the identity on the blocks

$$[\mathbb{1} f]_y := 2^4 \sum_{y'} \frac{1}{2^4} \delta_{y', y} f(y) = f(y) \quad (9)$$

according to

$$\nabla_\mu^{(+)} := \frac{1}{2} (U_\mu T_\mu^{(+)} - \mathbb{1}) , \quad \nabla_\mu^{(-)} := \frac{1}{2} (\mathbb{1} - T_\mu^{(-)} U_\mu^\dagger) \quad (10)$$

$$\Delta_\mu := \frac{1}{2} (\nabla_\mu^{(+)} - \nabla_\mu^{(-)}) = \frac{1}{4} (U_\mu T_\mu^{(+)} + T_\mu^{(-)} U_\mu^\dagger - 2 \mathbb{1}) . \quad (11)$$

We can recognize that the projections of the fermionic field

$$q_+ = P_+ q, \quad q_-^\dagger = P_- q \quad (12)$$

where

$$P_\pm = \frac{1}{2}(\mathbb{1} \otimes \mathbb{1} \mp \gamma_0 \gamma_5 \otimes t_5 t_0) \quad (13)$$

propagate forward/backward in time, and therefore describe particles/antiparticles respectively. Accordingly we introduce creation and annihilation operators $\hat{q}_\pm^\dagger, \hat{q}_\pm$. They are defined at one and the same time, so that in addition to spin and flavour, they depend on the spatial position only, denoted by boldface letters. They satisfy canonical anticommutation relations

$$\{(\hat{q}_\pm^\dagger)_{\mathbf{y}_1}^{a\alpha}, (\hat{q}_\pm)_{\mathbf{y}_2}^{\beta b}\} = \frac{1}{8} \delta_{\mathbf{y}_1 \mathbf{y}_2} P_\pm^{\beta b, \alpha a}, \quad \{(\hat{q}_\pm^\dagger)_{\mathbf{y}_1}^{a\alpha}, (\hat{q}_\mp)_{\mathbf{y}_2}^{\beta b}\} = 0. \quad (14)$$

As the factor $\frac{1}{8}$ accounts for the spatial volume of the blocks, the above anticommutation relations become canonical in the basis in which P_\pm are diagonal.

The transfer matrix corresponding to the flavour-Lagrangian (6) in the gauge $U_0 = \mathbb{1}$ is [14, 27]

$$\mathcal{T}_{t,t+1} = \exp(\hat{q}_- N_t \hat{q}_+^\dagger) \exp(2\mu \hat{n}_B) \exp(\hat{q}_- N_{t+1} \hat{q}_+). \quad (15)$$

In the above equation N_t is a matrix which depends on the time of the blocks only because it depends on the gauge link variables

$$N_t := N[U(t)], \quad (16)$$

and μ is the chemical potential

$$\hat{n}_B = 2^3 \sum_{\mathbf{y}} \left(\hat{q}_+^\dagger \hat{q}_+ - \hat{q}_-^\dagger \hat{q}_- \right)_{\mathbf{y}} \quad (17)$$

that we omitted for simplicity in the Lagrangian. By keeping into account the spatial volume factors

$$\hat{q}_- N_t \hat{q}_+ = 64 \sum_{\mathbf{y}', \mathbf{y}} (\hat{q}_-)_{\mathbf{y}'} (N_t)_{\mathbf{y}' \mathbf{y}} (\hat{q}_+)_{\mathbf{y}} \quad (18)$$

$$N_{\mathbf{y}' \mathbf{y}} = -2 \left\{ m (\gamma_0 \otimes \mathbb{1}) \mathbb{1}_{\mathbf{y}' \mathbf{y}} + \sum_{k=1}^3 (\gamma_0 \gamma_k \otimes \mathbb{1}) \left[P_k^{(-)} \nabla_k^{(+)} + P_k^{(+)} \nabla_k^{(-)} \right]_{\mathbf{y}' \mathbf{y}} \right\} \quad (19)$$

where

$$P_k^{(\pm)} = \frac{1}{2}(\mathbb{1} \otimes \mathbb{1} \pm \gamma_k \gamma_5 \otimes t_5 t_k) \quad (20)$$

and

$$\mathbb{1}_{\mathbf{y}'\mathbf{y}} := \frac{1}{8} \delta_{\mathbf{y}'\mathbf{y}}, \quad (T_k)_{\mathbf{y}'\mathbf{y}}^{(\pm)} := \frac{1}{8} \delta_{\mathbf{y}'\pm\hat{k},\mathbf{y}} \quad (21)$$

enter in the definitions of $\nabla_k^{(\pm)}$.

Notice that

$$q_{\pm}^{\dagger} N q_{\pm} = 0. \quad (22)$$

2.2 The spin basis

For the sake of later comparison we report the regularization of a Lagrangian in the spin basis. The gauge fields on the hypercubic lattice are denoted by $u_{\mu}(x)$ and transform according to

$$u_{\mu}(x) \rightarrow g_x u_{\mu}(x) g_{x+\hat{\mu}}^{\dagger}. \quad (23)$$

The Lagrangian in the spin basis is

$$\mathcal{L}_{\psi}(u) := m \bar{\psi}_x \psi_x + \frac{1}{2} \sum_{\mu=0}^3 \alpha_{x\mu} [\bar{\psi}_x u_{\mu}(x) \psi_{x+\hat{\mu}} - \bar{\psi}_{x+\mu} u_{\mu}^{\dagger}(x) \psi_x] \quad (24)$$

where the signs $\alpha_{x\mu}$ are defined for $\mu = 0, \dots, 3$ by

$$\alpha_{x\mu} := (-1)^{x_0 + \dots + x_{\mu-1}}. \quad (25)$$

There is no direct way of identifying forward and backward movers. This is the difficulty encountered in the construction of a transfer matrix in operator form for this Lagrangian. Indeed, as far as we know, such a construction has been achieved only after a reduction of the Lagrangian itself, in which the fermion fields and their conjugates live on odd and, respectively, even sites [11].

At the classical level, however, the fields in the spin and flavour basis are related according to

$$q_y^{\alpha a} = \frac{1}{8} \sum_{\eta} \Gamma_{\eta; \alpha a} \psi_{2y+\eta} \quad (26)$$

$$\bar{q}_y^{a\alpha} = \frac{1}{8} \sum_{\eta} \bar{\psi}_{2y+\eta} \Gamma_{\eta; a\alpha}^{\dagger} \quad (27)$$

where

$$\Gamma_{\eta} := \gamma_0^{\eta_0} \gamma_1^{\eta_1} \gamma_2^{\eta_2} \gamma_3^{\eta_3}. \quad (28)$$

The matrices Γ satisfy the relations

$$\frac{1}{4} \text{tr} (\Gamma_\eta^\dagger \Gamma_{\eta'}) = \delta_{\eta\eta'} \quad (29)$$

$$\frac{1}{4} \sum_\eta \Gamma_{\eta:b\beta}^\dagger \Gamma_{\eta:\alpha a} = \delta_{ba} \delta_{\beta\alpha} \quad (30)$$

that allow us to invert Eqs. (27)

$$\psi_{2y+\eta} = 2 \text{tr} (\Gamma_\eta^\dagger q_y) \quad (31)$$

$$\bar{\psi}_{2y+\eta} = 2 \text{tr} (\bar{q}_y \Gamma_\eta) . \quad (32)$$

We will use these relationships in order to derive an action and a transfer matrix in the spin basis from those in the flavour basis.

3 Transformation of the Lagrangian

In this Section we express the Lagrangian (6) in the spin basis using the transformations (27)

$$\sum_x \mathcal{L}'_\psi(U) := 2^4 \sum_y \mathcal{L}_q(U) . \quad (33)$$

While in the absence of gauge interaction \mathcal{L}'_ψ coincides with \mathcal{L}_ψ , reported in (24), we shall see that this does not occur, in general, in the presence of gauge fields.

The mass term of the action is proportional to

$$2^4 \sum_y \bar{q}_y q_y = \frac{1}{4} \sum_y \sum_\eta \sum_{\eta'} \bar{\psi}_{2y+\eta} \text{tr} (\Gamma_{\eta'}^\dagger \Gamma_\eta) \psi_{2y+\eta'} = \sum_x \bar{\psi}_x \psi_x . \quad (34)$$

In order to derive the kinetic term we shall use the relations

$$\sum_\alpha \gamma_\mu^{\alpha'\alpha} \Gamma_{\eta:\alpha a} = \delta_{0\eta\mu} \alpha_{\eta\mu} \Gamma_{\eta+\hat{\mu}:\alpha' a} + \delta_{1\eta\mu} \alpha_{\eta\mu} \Gamma_{\eta-\hat{\mu}:\alpha' a} \quad (35)$$

$$\sum_{\alpha,a} \gamma_5^{\alpha'\alpha} (t_5 t_\mu)^{a'a} \Gamma_{\eta:\alpha a} = -\delta_{0\eta\mu} \alpha_{\eta\mu} \Gamma_{\eta+\hat{\mu}:\alpha' a'} + \delta_{1\eta\mu} \alpha_{\eta\mu} \Gamma_{\eta-\hat{\mu}:\alpha' a'} \quad (36)$$

From the definition (28) soon follow the relation (35) and

$$\Gamma_\eta \gamma_\mu = (-1)^{\eta_0+\eta_1+\eta_2+\eta_3} (-1)^{\eta_\mu} \gamma_\mu \Gamma_\eta \quad (37)$$

so that

$$\Gamma_\eta \gamma_5 = (-1)^{\eta_0+\eta_1+\eta_2+\eta_3} \gamma_5 \Gamma_\eta \quad (38)$$

and therefore

$$\sum_{\alpha,a} \gamma_5^{\alpha'\alpha} (t_5 t_\mu)^{a'a} \Gamma_{\eta:\alpha a} = (\gamma_5 \Gamma_\eta \gamma_\mu \gamma_5)_{\alpha'a'} = -(\gamma_5 \Gamma_\eta \gamma_5 \gamma_\mu)_{\alpha'a'} = -(-1)^{\eta_\mu} (\gamma_\mu \Gamma_\eta)_{\alpha'a'} \quad (39)$$

which together with (35) implies the relation (36).

The kinetic term is proportional to

$$\begin{aligned} \frac{16}{4} \sum_y \sum_\mu \{ \bar{q}_y (\gamma_\mu \otimes 1) [U_\mu(y) q_{y+\hat{\mu}} - U_\mu^\dagger(y - \hat{\mu}) q_{y-\hat{\mu}}] \\ - \bar{q}_y (\gamma_5 \otimes t_5) t_\mu [U_\mu(y) q_{y+\hat{\mu}} + U_\mu^\dagger(y - \hat{\mu}) q_{y-\hat{\mu}} - 2q_y] \} \end{aligned} \quad (40)$$

that is

$$\begin{aligned} \frac{1}{16} \sum_y \sum_\mu \sum_{\eta,\eta'} \sum_{\alpha,\alpha',a,a'} \bar{\psi}_{2y+\eta'} \Gamma_{\eta':\alpha'\alpha'}^\dagger \left[U_\mu(y) \left(\gamma_\mu^{\alpha'\alpha} \delta^{a'a} - \gamma_5^{\alpha'\alpha} (t_5 t_\mu)^{a'a} \right) \Gamma_{\eta:\alpha a} \psi_{2y+2\hat{\mu}+\eta} \right. \\ \left. - U_\mu^\dagger(y - \hat{\mu}) \left(\gamma_\mu^{\alpha'\alpha} \delta^{a'a} + \gamma_5^{\alpha'\alpha} (t_5 t_\mu)^{a'a} \right) \Gamma_{\eta:\alpha a} \psi_{2y-2\hat{\mu}+\eta} + 2\gamma_5^{\alpha'\alpha} (t_5 t_\mu)^{a'a} \Gamma_{\eta:\alpha a} \psi_{2y+\eta} \right] \end{aligned} \quad (41)$$

which is because of (35) and (36)

$$\begin{aligned} \frac{1}{8} \sum_y \sum_\mu \sum_{\eta,\eta'} \sum_{\alpha,\alpha',a,a'} \bar{\psi}_{2y+\eta'} \Gamma_{\eta':\alpha'\alpha'}^\dagger \alpha_{\eta\mu} \left[U_\mu(y) \delta_{0\eta_\mu} \Gamma_{\eta+\hat{\mu}:\alpha'a'} \psi_{2y+2\hat{\mu}+\eta} \right. \\ \left. - U_\mu^\dagger(y - \hat{\mu}) \delta_{1\eta_\mu} \Gamma_{\eta-\hat{\mu}:\alpha'a'} \psi_{2y-2\hat{\mu}+\eta} + (-\delta_{0\eta_\mu} \Gamma_{\eta+\hat{\mu}:\alpha'a'} + \delta_{1\eta_\mu} \Gamma_{\eta-\hat{\mu}:\alpha'a'}) \psi_{2y+\eta} \right] \end{aligned} \quad (42)$$

and performing the trace on spinor and flavour indices (30)

$$\begin{aligned} \frac{1}{2} \sum_y \sum_\mu \sum_{\eta,\eta'} \bar{\psi}_{2y+\eta'} \alpha_{\eta\mu} \left[U_\mu(y) \delta_{0\eta_\mu} \delta_{\eta',\eta+\hat{\mu}} \psi_{2y+2\hat{\mu}+\eta} \right. \\ \left. - U_\mu^\dagger(y - \hat{\mu}) \delta_{1\eta_\mu} \delta_{\eta',\eta-\hat{\mu}} \psi_{2y-2\hat{\mu}+\eta} + (-\delta_{0\eta_\mu} \delta_{\eta',\eta+\hat{\mu}} + \delta_{1\eta_\mu} \delta_{\eta',\eta-\hat{\mu}}) \psi_{2y+\eta} \right] \end{aligned} \quad (43)$$

and performing the sum over η'

$$\begin{aligned} \frac{1}{2} \sum_y \sum_\eta \sum_\mu \alpha_{\eta\mu} \left[\delta_{0\eta_\mu} \bar{\psi}_{2y+\eta+\hat{\mu}} U_\mu(y) \psi_{2(y+\hat{\mu})+\eta} + \delta_{1\eta_\mu} \bar{\psi}_{2y+\eta-\hat{\mu}} \psi_{2y+\eta} \right. \\ \left. - \delta_{1\eta_\mu} \bar{\psi}_{2y+\eta-\hat{\mu}} U_\mu^\dagger(y - \hat{\mu}) \psi_{2(y-\hat{\mu})+\eta} - \delta_{0\eta_\mu} \bar{\psi}_{2y+\eta+\hat{\mu}} \psi_{2y+\eta} \right]. \end{aligned} \quad (44)$$

Remark that if we increase the component x_μ of a site x we jump on block different from that of x if x_μ is odd. This is the case when $x = 2y + \eta + \hat{\mu}$ and $\eta_\mu = 0$, but not when $x = 2y + \eta - \hat{\mu}$ and $\eta_\mu = 1$. Similarly, if we decrease x_μ we

jump on a different block only when x_μ is even. This is the case when $x = 2y + \eta - \hat{\mu}$ and $\eta_\mu = 1$, but not when $x = 2y + \eta + \hat{\mu}$ and $\eta_\mu = 0$. And that, if $x = 2y + \eta$ then

$$\alpha_{\eta\mu} = \alpha_{x\mu} . \quad (45)$$

Then the kinetic term has the form as that of $\mathcal{L}_\psi(u')$ where

$$u'_\mu(x) = \begin{cases} U_\mu(y) & \text{for } x = 2y + \eta \text{ and } \eta_\mu = 1 \\ \mathbb{1} & \text{elsewhere} \end{cases} \quad (46)$$

that is the gauge field couples only sites which belong to different blocks.

In conclusion

$$\mathcal{L}'_\psi(u') = m \bar{\psi}_x \psi_x + \frac{1}{2} \sum_{\mu=0}^3 \alpha_{x\mu} [\bar{\psi}_x u'_\mu(x) \psi_{x+\hat{\mu}} - \bar{\psi}_{x+\mu} u'^\dagger_\mu(x) \psi_x] . \quad (47)$$

We have the constraint, however, that the fermion fields within a block should all transform in the same way under gauge transformations. One might think that we could relax this constraint by a different transformation from the spin to the flavour basis

$$\begin{aligned} q_y^{\alpha a} &= \frac{1}{8} \sum_{\eta} \Gamma_{\eta; \alpha a} \mathcal{C}_{2y+\eta} \psi_{2y+\eta} \\ \bar{q}_y^{a\alpha} &= \frac{1}{8} \sum_{\eta} \bar{\psi}_{2y+\eta} \mathcal{C}_{2y+\eta}^\dagger \Gamma_{\eta; \alpha a}^\dagger . \end{aligned} \quad (48)$$

Such a generalization, however, is only apparent, because the curvature for the plaquettes with all the vertices within one and the same block vanishes. Indeed, such a generalization, as the particular ones chosen for example in [9, Eq. (35)], [27, Eq. (56)] is a pure-gauge transformation of (27).

We conclude that, in the presence of a *generic* gauge-field configuration, the Lagrangian in the spin basis $\mathcal{L}'_\psi(u)$ and that in the flavor basis $\mathcal{L}_q(U)$ are not equivalent.

The transformed Lagrangian $\mathcal{L}'_\psi(u')$ could also be regarded, in the spirit of the previous quoted attempt [11], as a modification of $\mathcal{L}_\psi(u)$, defined in (24), for which a transfer matrix can be constructed.

The above construction refers to the case of vanishing chemical potential. Its inclusion is, however, straightforward [27]. We only note that, at variance with respect to the coupling with gauge fields, the chemical potential can be attached to all links in the transformed Lagrangian $\mathcal{L}'_\psi(u')$, provided its value be half the one in the flavour basis.

4 Transformation of transfer matrix and coherent states

As a first step we must transform creation-annihilation operators from the flavour to the spin basis. To this end we must determine the expressions of the fields q_{\pm} in the spin basis

$$(q_+)_{\mathbf{y}} = P_+ \frac{1}{8} \sum_{\eta} \Gamma_{\eta} \psi_{2\mathbf{y}+\eta} \quad (q_-^{\dagger})_{\mathbf{y}} = P_- \frac{1}{8} \sum_{\eta} \Gamma_{\eta} \psi_{2\mathbf{y}+\eta}. \quad (49)$$

Using the relation (39), we find

$$P_+ \Gamma_{\eta} = \delta_{0\eta_0} \Gamma_{\eta}, \quad P_- \Gamma_{\eta} = \delta_{1\eta_0} \Gamma_{\eta} \quad (50)$$

and similar relations hold for Γ^{\dagger} .

We therefore have

$$(q_+)_{\mathbf{y}} = \frac{1}{8} \sum_{\eta} \delta_{0\eta_0} \Gamma_{\eta} \psi_{2\mathbf{y}+\eta}, \quad (q_-^{\dagger})_{\mathbf{y}} = \frac{1}{8} \sum_{\eta} \delta_{1\eta_0} \Gamma_{\eta} \psi_{2\mathbf{y}+\eta}. \quad (51)$$

Next we define the operators corresponding to the ψ -fields according to

$$(\hat{q}_+)_{\mathbf{y}} = \frac{1}{8} \sum_{\eta} \delta_{0\eta_0} \Gamma_{\eta} \hat{\psi}_{2\mathbf{y}+\eta}, \quad (\hat{q}_-^{\dagger})_{\mathbf{y}} = \frac{1}{8} \sum_{\eta} \delta_{1\eta_0} \Gamma_{\eta} \hat{\psi}_{2\mathbf{y}+\eta} \quad (52)$$

and assume that

$$\{\hat{\psi}_{2\mathbf{y}'+\eta'}^{\dagger}, \hat{\psi}_{2\mathbf{y}+\eta}\} = 2 \delta_{\mathbf{y}'\mathbf{y}} \delta_{\eta'\eta}. \quad (53)$$

This is obviously consistent with the second set of equations in (14). Consistency with the first set requires that

$$\frac{1}{64} \sum_{\eta, \eta'} \delta_{\sigma\eta_0} \delta_{\tau\eta'_0} \Gamma_{\eta':b\beta}^{\dagger} \Gamma_{\eta:\alpha a} \{\hat{\psi}_{2\mathbf{y}'+\eta'}^{\dagger}, \hat{\psi}_{2\mathbf{y}+\eta}\} = \frac{1}{32} \delta_{\mathbf{y}'\mathbf{y}} \delta_{\sigma\tau} \sum_{\eta} \delta_{\sigma\eta_0} \Gamma_{\eta:b\beta}^{\dagger} \Gamma_{\eta:\alpha a} \quad (54)$$

$$= \frac{1}{8} \delta_{\mathbf{y}'\mathbf{y}} \delta_{\sigma\tau} P_{\pm}^{\alpha a, \beta b} \quad (55)$$

where $\sigma = 0, 1$ respectively when the index of the projector is $+$ or $-$. The second equality follows from the equations

$$\sum_{a', \alpha'} P_{\pm}^{\alpha a, \alpha' a'} \frac{1}{4} \sum_{\eta} \Gamma_{\eta:b\beta}^{\dagger} \Gamma_{\eta:\alpha' a'} = \frac{1}{4} \sum_{\eta} \delta_{\sigma\eta_0} \Gamma_{\eta:b\beta}^{\dagger} \Gamma_{\eta:\alpha a} = P_{\pm}^{\alpha a, \beta b} \quad (56)$$

that can be proven using (30) and (50).

Some comments about our results are in order. We see that the temporal component η_0 of the fields in the spinor basis corresponds to the \pm projection of the field in the flavor basis. The 8 Dirac-taste degrees of freedom of particles/antiparticles are spread on the 8 sites of the even/odd time slice in the corresponding block. In this connection, looking at Eq.(53), η_0 can be regarded as a quantum number. But this quantum number changes when time increases by one unit in the original lattice, so that, unlike the q_{\pm} projections, the fields $\psi_{2\mathbf{y}+\eta}$ with η_0 respectively 1 or 0 cannot be identified as forward/backward movers. Changing time we change a particle into the hole of an antiparticle.

4.1 Transfer matrix

We first transform the baryon number

$$\hat{n}_B = 2^3 \sum_{\mathbf{y}} \left(\hat{q}_+^\dagger \hat{q}_+ - \hat{q}_-^\dagger \hat{q}_- \right)_{\mathbf{y}} = \frac{1}{2} \sum_{\mathbf{y}, \eta} \left[\left(\hat{\psi}^\dagger \hat{\psi} \right)_{2\mathbf{y}+\eta} \delta_{0\eta_0} - \left(\hat{\psi} \hat{\psi}^\dagger \right)_{2\mathbf{y}+\eta} \delta_{1\eta_0} \right] \quad (57)$$

$$= \frac{1}{2} \sum_{\mathbf{x}} \left[\left(\hat{\psi}^\dagger \hat{\psi} \right)_{\mathbf{x}0} - \left(\hat{\psi} \hat{\psi}^\dagger \right)_{\mathbf{x}1} \right] \quad (58)$$

where we re-label the operators $\hat{\psi}$ with the spatial coordinates

$$\mathbf{x} = 2\mathbf{y} + \boldsymbol{\eta} \quad (59)$$

and η_0 and made the identifications

$$\hat{\psi}_{\mathbf{x}0} := \hat{\psi}_{2\mathbf{y}+(0,\boldsymbol{\eta})}, \quad \hat{\psi}_{\mathbf{x}1} := \hat{\psi}_{2\mathbf{y}+(1,\boldsymbol{\eta})}^\dagger \quad (60)$$

in agreement with the relations (52) which show that when $\eta_0 = 1$ the operator $\hat{\psi}_{2\mathbf{y}+\eta}$ is a creation operator.

In this notation the commutation relations (53) become

$$\{ \hat{\psi}_{\mathbf{x}'\eta'_0}^\dagger, \hat{\psi}_{\mathbf{x}\eta_0} \} = 2 \delta_{\mathbf{x}'\mathbf{x}} \delta_{\eta'_0\eta_0}. \quad (61)$$

Next we must determine a matrix N'_t such that

$$64 \sum_{\mathbf{y}', \mathbf{y}} (\hat{q}_-^\dagger)_{\mathbf{y}'} (N_t)_{\mathbf{y}'\mathbf{y}} (\hat{q}_+)_{\mathbf{y}} = \sum_{\mathbf{y}', \mathbf{y}} \sum_{\eta', \eta} \hat{\psi}_{2\mathbf{y}'+\eta'}^\dagger \text{tr} \left(\Gamma_{\eta'}^\dagger (N_t)_{\mathbf{y}'\mathbf{y}} P_+ \Gamma_{\eta} \right) \hat{\psi}_{2\mathbf{y}+\eta} \quad (62)$$

$$= \sum_{\mathbf{y}', \mathbf{y}} \sum_{\eta', \eta} \hat{\psi}_{2\mathbf{y}'+\eta'}^\dagger (N'_t)_{\mathbf{y}'\eta', \mathbf{y}\eta} \hat{\psi}_{2\mathbf{y}+\eta}. \quad (63)$$

In the above equation color taste and Dirac indices have been omitted. We observe that

$$(\gamma_0 \gamma_k \otimes \mathbb{1}) P_k^{(\pm)} P_+ \Gamma_\eta = \frac{1}{2} \delta_{0\eta_0} [(\gamma_0 \gamma_k \otimes \mathbb{1}) \pm (\gamma_0 \gamma_5 \otimes t_5 t_k)] \Gamma_\eta \quad (64)$$

$$= \delta_{0\eta_0} \alpha_{\eta k} \frac{1 \mp (-1)^{\eta_k}}{2} (\delta_{0\eta_k} \Gamma_{\eta+\hat{0}+\hat{k}} + \delta_{1\eta_k} \Gamma_{\eta+\hat{0}-\hat{k}}) \quad (65)$$

and

$$\Gamma_{\eta'}^\dagger (\gamma_0 \otimes \mathbb{1}) P_+ \Gamma_\eta = \delta_{0\eta_0} \delta_{1\eta'_0} \Gamma_{\eta'}^\dagger \Gamma_\eta, \quad (66)$$

so that

$$\text{tr} \left[\Gamma_{\eta'}^\dagger (\gamma_0 \gamma_k \otimes \mathbb{1}) P_k^{(\pm)} P_+ \Gamma_\eta \right] = 4 \delta_{0\eta_0} \delta_{1\eta'_0} \alpha_{\eta k} \frac{1 \mp (-1)^{\eta_k}}{2} (\delta_{0\eta_k} \delta_{\eta', \eta+\hat{k}} + \delta_{1\eta_k} \delta_{\eta', \eta-\hat{k}}) \quad (67)$$

and

$$\text{tr} \left[\Gamma_{\eta'}^\dagger (\gamma_0 \otimes \mathbb{1}) P_+ \Gamma_\eta \right] = 4 \delta_{0\eta_0} \delta_{1\eta'_0} \delta_{\eta' \eta}. \quad (68)$$

Finally we get the transformed N -matrix

$$\begin{aligned} (N')_{\mathbf{y}' \eta', \mathbf{y} \eta} &= -8 \delta_{0\eta_0} \delta_{1\eta'_0} \left[m \delta_{\eta' \eta} \mathbb{1}_{\mathbf{y}' \mathbf{y}} + \sum_{\mu=1}^3 \alpha_{\eta \mu} (\delta_{0\eta_\mu} \delta_{\eta', \eta+\hat{\mu}} \nabla_\mu^{(+)} + \delta_{1\eta_\mu} \delta_{\eta', \eta-\hat{\mu}} \nabla_\mu^{(-)})_{\mathbf{y}' \mathbf{y}} \right] \\ &= -\delta_{0\eta_0} \delta_{1\eta'_0} \left\{ m \delta_{\eta' \eta} \delta_{\mathbf{y}' \mathbf{y}} + \frac{1}{2} \sum_{\mu=1}^3 \alpha_{\eta \mu} [(-\delta_{0\eta_\mu} \delta_{\eta', \eta+\hat{\mu}} + \delta_{1\eta_\mu} \delta_{\eta', \eta-\hat{\mu}}) \delta_{\mathbf{y}' \mathbf{y}} \right. \\ &\quad \left. + \delta_{0\eta_\mu} \delta_{\eta', \eta+\hat{\mu}} U_\mu(\mathbf{y}') \delta_{\mathbf{y}, \mathbf{y}'+\hat{\mu}} - \delta_{1\eta_\mu} \delta_{\eta', \eta-\hat{\mu}} U_\mu^\dagger(\mathbf{y}) \delta_{\mathbf{y}, \mathbf{y}'-\hat{\mu}}] \right\}. \quad (69) \end{aligned}$$

Notice that the terms that involve the gauge variables refer to sites belonging to different blocks, while in the other terms the sites belong to the same blocks. The same operator can be re-labelled by using the coordinates \mathbf{x} and η_0 , then

$$(N')_{\mathbf{x}' \eta'_0, \mathbf{x} \eta_0} = -\delta_{0\eta_0} \delta_{1\eta'_0} \left\{ m \delta_{\mathbf{x}' \mathbf{x}} + \frac{1}{2} \sum_{\mu=1}^3 \alpha_{\mathbf{x} \mu} [\delta_{\mathbf{x}', \mathbf{x}-\hat{\mu}} u'_\mu(\mathbf{x}') - \delta_{\mathbf{x}', \mathbf{x}+\hat{\mu}} u_\mu^\dagger(\mathbf{x})] \right\} \quad (70)$$

where the values $\eta_\mu = 0, 1$ simply control the presence of the gauge field according to the definition of u' given in (46).

In conclusion

$$\hat{q}_- N_t \hat{q}_+ = \hat{\psi}_1 N'_t \hat{\psi}_0. \quad (71)$$

It should not be necessary to repeat that the expression of the transfer matrix so obtained is positive definite and performs time translations by two lattice spacings.

4.2 Coherent states

In order to complete our analysis we perform the transformation also on the coherent states. This will enable us to make, as a crosscheck, the derivation of the Lagrangian (47) starting from the transfer matrix.

Let

$$|\alpha, \beta\rangle := \exp \left\{ -2^3 \sum_{\mathbf{y}} \sum_{\gamma, c} [\alpha_{\mathbf{y}}^{\gamma c} (\hat{q}_+^\dagger)_{\mathbf{y}}^{c\gamma} + \beta_{\mathbf{y}}^{c\gamma} (\hat{q}_-^\dagger)_{\mathbf{y}}^{\gamma c}] \right\} |0\rangle \quad (72)$$

be a coherent state in the flavour basis, where $\alpha_{\mathbf{y}}^{\gamma c}$ and $\beta_{\mathbf{y}}^{\gamma c}$ are Grassmann variables, such that

$$(\hat{q}_+)^{\gamma c}_{\mathbf{y}} |\alpha, \beta\rangle = \alpha_{\mathbf{y}}^{\gamma c} |\alpha, \beta\rangle, \quad (\hat{q}_-)^{c\gamma}_{\mathbf{y}} |\alpha, \beta\rangle = \beta_{\mathbf{y}}^{c\gamma} |\alpha, \beta\rangle \quad (73)$$

Now

$$2^3 \sum_{\mathbf{y}} \sum_{\gamma, c} \alpha_{\mathbf{y}}^{\gamma c} (\hat{q}_+^\dagger)_{\mathbf{y}}^{c\gamma} = \sum_{\mathbf{y}, \eta} \text{tr} (\Gamma_{\eta}^\dagger \alpha_{\mathbf{y}}) \delta_{0\eta_0} \hat{\psi}_{2\mathbf{y}+\eta}^\dagger \quad (74)$$

$$2^3 \sum_{\mathbf{y}} \sum_{\gamma, c} \beta_{\mathbf{y}}^{c\gamma} (\hat{q}_-^\dagger)_{\mathbf{y}}^{\gamma c} = \sum_{\mathbf{y}, \eta} \text{tr} (\beta_{\mathbf{y}} \Gamma_{\eta}) \delta_{1\eta_0} \hat{\psi}_{2\mathbf{y}+\eta} \quad (75)$$

and therefore, because of the anti-commutation relations (61)

$$\hat{\psi}_{\mathbf{x}0} |\alpha, \beta\rangle = \sum_{\eta_0} \hat{\psi}_{2\mathbf{y}+\eta} \delta_{0\eta_0} |\alpha, \beta\rangle = 2 \text{tr} (\Gamma_{(0, \boldsymbol{\eta})}^\dagger \alpha_{\mathbf{y}}) |\alpha, \beta\rangle \quad (76)$$

$$\hat{\psi}_{\mathbf{x}1} |\alpha, \beta\rangle = \sum_{\eta_0} \hat{\psi}_{2\mathbf{y}+\eta}^\dagger \delta_{1\eta_0} |\alpha, \beta\rangle = 2 \text{tr} (\beta_{\mathbf{y}} \Gamma_{(1, \boldsymbol{\eta})}) |\alpha, \beta\rangle. \quad (77)$$

This means that we can define

$$\alpha'_{\mathbf{x}} := 2 \text{tr} (\Gamma_{(0, \boldsymbol{\eta})}^\dagger \alpha_{\mathbf{y}}), \quad \beta'_{\mathbf{x}} := 2 \text{tr} (\beta_{\mathbf{y}} \Gamma_{(1, \boldsymbol{\eta})}) \quad (78)$$

and re-write

$$|\alpha, \beta\rangle = \exp \left[-\frac{1}{2} \sum_{\mathbf{x}} (\alpha'_{\mathbf{x}} \hat{\psi}_{\mathbf{x}0}^\dagger + \beta'_{\mathbf{x}} \hat{\psi}_{\mathbf{x}1}^\dagger) \right] |0\rangle. \quad (79)$$

Notice that the Grassmann variables α, β and α' as well are defined at even times. The variable β' instead, because of the matrix $\Gamma_{(1, \boldsymbol{\eta})}$ in its definition, must be considered attached at odd times. This is confirmed by the evaluation of the partition function using the transformed transfer matrix and coherent states. After the identifications

$$\begin{aligned} \overline{\psi}_{2x_0} &= (\alpha'_{2x_0})^*, \quad \psi_{2x_0} = (\beta'_{2x_0+1})^* \\ \overline{\psi}_{2x_0+1} &= \beta'_{2x_0+3}, \quad \psi_{2x_0+1} = \alpha'_{2x_0+2} \end{aligned} \quad (80)$$

we get the Lagrangian (47).

5 Conclusion

Numerical simulations with Kogut-Susskind fermions are faster in the spin basis than in the flavor basis. Such calculations are usually performed in the lagrangian formulation, but we are interested in numerical simulations in the framework of the nilpotency expansion, that makes use of the transfer matrix. So we need an expression of the transfer matrix in the spin basis. In any case the knowledge of a positive definite transfer matrix in the spin basis is *per se* interesting being related to the unitarity of the theory.

We found in the literature essentially two formulations of the transfer matrix in the spin basis. In the first one the Lagrangian is reduced by defining fermion fields and their conjugates at the odd, respectively even sites, and a transfer matrix is constructed that performs time translations by 2 lattice spacings [11, 12]. The fermion determinant even at vanishing chemical potential, is, however, not positive definite, which makes this way less suitable to numerical simulations.

In the second formulation [11], a positive definite transfer matrix, called T^2 , was defined that also performs time translations by 2 lattice spacings. As a consequence the corresponding Fock space must be constructed on blocks. The explicit construction of such Fock space, however, is not given.

If the Fock space is associated to a block, we can get the transfer matrix in the spin basis by a unitary transformation from that in the flavor basis, whose expression, together with the construction of the Fock space, are known. The transfer matrix in the flavor basis is expressed in terms of a matrix N , and the transformed matrix is given in terms of the matrix N' , given explicitly in (70). In order to do numerical simulations in the nilpotency expansion all we need is to replace everywhere in the equations of the nilpotency expansion N by N' and remember that the gauge fields are now defined on blocks.

It would be now natural to compare our result with the expression of the previously derived transfer matrix [11]. One might expect that such a comparison should provide the definition of the Fock space in the latter. Unfortunately this is not the case. The transfer matrix of [11] cannot be related to ours in a simple way, the most remarkable differences being that there is no requirement concerning the gauge variables which remain defined on the links of the original lattice, and creation and annihilation operators appear not only in exponential form but also as powers.

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